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INVERSE PROBLEMS FOR EVOLUTION EQUATIONS OF THE INTEGRODIFFERENTIAL TYPE*

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The problem of determining the relaxation kernel using a solution of the inverse problem for an evolution equation which describes the distortion of the profile of an individual wave is analysed. The general case when the kernel is specified in parametric form, and the special case of an exponential kernel, are considered. The non-linear inverse problem is solved using the gradient method. For a linear equation the possibility of using the Laplace transformation method is pointed out.

1. Suppose we are given the following equation of motion:

$$\rho U_{TT}(T,X)=G(0)\,U_{XX}(T,X)+\frac{\partial}{\partial X}\int_0^\infty G(s)\,U_X(T-s,X)\,ds \tag{1.1}$$

$$U(0,X)=U_T(0,X)=0,\quad U_X(T,0)=\varphi_0(T),\ \lim_{X\to\infty}U(T,X)=0$$

Here U is the transposition, ρ is the density, X is the Lagrange coordinate and T is the time. The inverse problem for Eq.(1.1) consists of the following. The measurement $U_x(T,\bar{X})=\varphi(T,\bar{X})$ is specified at the point $X=\bar{X}$, and it is required to determine the kernel $G(s)$ using the function $\varphi(T,\bar{X})$.

For a stationary monochromatic wave of frequency ω the following method is most effective for solving the inverse problem. The phase velocity $c(\omega)$ and the attenuation coefficient are determined experimentally and the kernel $G(s)$ /1/ is established using Fourier's transformation. The solution of the problem is quite complicated in the non-linear formulation. In this case it is feasible to break down the wave process into separate waves, which leads to a description of the evolution of the separate waves by the respective evolution equations /2/. In the general case the one-dimensional evolution equation of the first approximation for a longitudinal wave has the form

$$u_t+a_{01}uu_x-\frac{\partial}{\partial x}\int_0^x u_z(t,z)\,K(x-z)\,dz=0 \tag{1.2}$$

$$x=c_0T-X,\quad t=\varepsilon X$$

where c_0 is the velocity of the longitudinal wave, ε is some small parameter and $u(t,x)$ is the first approximation of the particle velocity (a transition to the deformation u_x is possible) and $K(x)$ is the kernel; the coefficient $a_{01}=\mathrm{const}$ determines the effect of the geometric and physical non-linearities /3/. Details of the transition from the second-order equation of type (1.1) or from the system of higher order to an evolution equation of type (1.2) are given in /2/.

We shall add the following conditions to Eq.(1.2):

$$u(0,x)=u_0(x),\ u(t,0)=0 \tag{1.3}$$

We will assume that the kernel $K(x)$ is such that Eq.(1.2) has a unique solution $u(t,x)$, satisfying condition (1.3). The inverse problem for Eq.(1.2) consists of determining the kernel $K(x)$ from the condition $u(\bar{t},x)=u_1(x)$, where $t=\bar{t}$ is a fixed instant of time. It is well-known that, generally speaking, inverse problems are ill-posed. If we assume that the unknown kernel $K(x)$ belongs to a specified compact set, then the following is known /4/. If the relation between the measurement $u_1(x)$ and the kernel $K(x)$ is one-to-one and continuous (in the defined metric), then the above inverse problem is Tikhonov well-posed. The validity of these conditions for a linear problem is established below. In the non-linear formulation the required properties strongly depend on the properties of the class of permissible kernels $K(x)$.

The solution of the inverse problem (1.2), (1.3) has a number of advantages compared with the problem of type (1.1): the inverse problem of the evolution equation (1.2) with initial conditions has been investigated more than the inverse problem (1.1) with a boundary mode /4/; the direct problem of solving Eq.(1.2) /5, 6/ which enables us to determine the

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distortion of the separate wave, corresponds very well to the possibilities of experimental technique; the order of the evolution equation is lower than that of the basic equation.

Since equations of the (1.2) type are derived using an asymptotic method, the inverse problems of determining their coefficients or kernel in its own physical meaning can be termed asymptotic. Some possibilities of solving the inverse problem for Eq.(1.2) in both the linear and non-linear formulations with the arbitrary function $u_0(x)$ are shown below. This enables us to use the proposed technique in pulse acousto-diagnostics.

2. First consider the parametric case when the kernel is specified in the form $K(x, \alpha)$, where α is an unknown parameter.

Consider the equation

$$u_t + a_{01}uu_x - \frac{\partial}{\partial x}\int_0^x K(x-z, \alpha)\, u_z dz = 0 \qquad (2.1)$$

$$u(0, x) = u_0(x), \quad u(t, 0) = 0$$

We will assume that the boundary and initial conditions agree, i.e. $u_0(0) = 0$. Suppose $\alpha = (\alpha_1, \ldots, \alpha_n) \in A$, where A is a closed, bounded and convex set in R_n. Suppose the function $K(x, \alpha)$ is such that for any $\alpha \in A$ the unique solution $u = u(t, x) = u(t, x; \alpha)$ of Eq.(2.1) exists and suppose $u(t, x; \alpha)$, $\partial u(t, x; \alpha)/\partial\alpha$, $K(x, \alpha)$ and $\partial K(x, \alpha)/\partial\alpha$ depend continuously on the parameter α. We will assume that if $\alpha_m \to \alpha$, then $u_m(t, x; \alpha_m)$ converges uniformly together with the derivatives to the function $u(t, x; \alpha)$ and its derivatives, respectively.

Suppose there is the possibility of estimating (measuring) the solution of Eq.(2.1) when $t = \bar{t}$. As a result of the measurement we will obtain the approximate value $g(x)$ of the exact solution, i.e.

$$u_1(x) = u(\bar{t}, x; \alpha) = g(x) + \varepsilon(x), \quad 0 \leqslant x \leqslant \xi \leqslant \infty \qquad (2.2)$$

Using this information (the function $g(x)$) it is required to determine the actual value α^* of the parameter $\alpha \in A$.

If the exact form of the solution $u(t, x; \alpha)$ is known, we have a problem of non-linear regression. But since the exact form of the solution is unknown, we are obliged to use other methods. We will consider the problem of determining the parameter α using the observed value $g(x)$ as an optimization problem, i.e. we will take $\alpha = \bar{\alpha}$ as the actual value of the parameter, such that $u(\bar{t}, x; \bar{\alpha})$ corresponds in the best way (in the root-mean-square sense) to the observed value $g(x)$.

Suppose

$$J(\alpha) = \int_0^\xi [u(\bar{t}, x; \alpha) - g(x)]^2 dx \qquad (2.3)$$

The problem of identifying the parameter α consists of the following: to find $\bar{\alpha} \in A$, such that $\min_\alpha J(\alpha) = J(\bar{\alpha})$. Since A is a closed and bounded set, $\bar{\alpha}$ exists and the problem of determining the parameter $\bar{\alpha}$ using the observations is correct. Nevertheless, if the number of components $\alpha_1, \ldots, \alpha_n$ is large, the problem of determining them becomes ill-posed and we must formulate the problem of choosing the number of parameters. Usually the number of parameters is chosen to be minimal under the condition that the calculation data is comparable with the measurement data. Specifying the error of measurement ε, we need to require that $|u(\bar{t}, x; \alpha^*) - g(x)| \leqslant \varepsilon$, where α^* are the values, calculated with respect to $g(x)$, of the parameter $\alpha = (\alpha_1, \ldots, \alpha_n)$.

To find the solution $\bar{\alpha}$ we must resort to different numerical methods which enable us to construct the minimizing sequence $\{\alpha^k\}$, $\alpha^k \to \bar{\alpha}$. In most of these methods the gradient of the function $J(\alpha)$ is used.

We shall describe, for example, the gradient projection method.

Suppose

$$\operatorname{grad} J(\alpha) = (\partial J/\partial\alpha_1, \ldots, \partial J/\partial\alpha_n) \qquad (2.4)$$

and $P_A(\alpha)$ is a projection of the element $\alpha \in R_n$ on to the convex space A. We will construct the sequence α^k using the rule

$$\alpha^{k+1} = P_A(\alpha^k - t_k \operatorname{grad} J(\alpha^k)), \quad k = 0, 1, \ldots \qquad (2.5)$$

where t_k is a positive quantity. The conditions of convergence of the sequence α^k to the local minimum of the function $J(\alpha)$ and the method of choosing the step t_k can be found, for example, in /7/.

On the basis of Eq.(2.1) we will introduce an expression for the gradient, i.e. we will find the function (2.4). A similar method of calculating the gradient using the conjugate equation was used, for example in /8/.

3. We shall use $u^\Delta = u(t, x; \alpha + \Delta\alpha)$ to denote the solution of Eq.(2.1) by replacing α by $\alpha + \Delta\alpha = (\alpha_1 + \Delta\alpha_1, \ldots, \alpha_n + \Delta\alpha_n)$. Then

$$(u^\Delta - u)_t + a_{01}(u^\Delta - u)\, u_x^\Delta + a_{01}u\,(u^\Delta - u)_x - \qquad (3.1)$$

$$\frac{\partial}{\partial x}\int_0^x K(x-z,\alpha+\Delta\alpha)(u^\Delta-u)_z\,dz -$$
$$\frac{\partial}{\partial x}\int_0^x [K(x-z,\alpha+\Delta\alpha)-K(x-z,\alpha)]\,u_z\,dz = 0$$
$$u^\Delta(0,x)-u(0,x)=0,\quad u^\Delta(t,0)-u(t,0)=0$$

We shall set $v_i = v_i(t,x;\alpha) = \partial u(t,x;\alpha)/\partial\alpha_i$. We divide both sides of Eq.(3.1) by $\Delta\alpha_i$, then as $\Delta\alpha\to 0$ we obtain

$$(v_i)_t + a_{01}(uv_i)_x - \frac{\partial}{\partial x}\int_0^x K(x-z,\alpha)(v_i)_z\,dz = \frac{\partial}{\partial x}\int_0^x \frac{\partial K(x-z,\alpha)}{\partial\alpha_i}\,u_z\,dz,\quad v_i(0,x)=v_i(t,0)=0 \tag{3.2}$$

$i = 1,\ldots,n$

It is obvious that

$$\frac{\partial J(\alpha)}{\partial\alpha_i} = 2\int_0^\xi [u(\bar t,x;\alpha)-g(x)]\,v_i(\bar t,x;\alpha)\,dx \tag{3.3}$$

To transform this expression we will introduce the following equation which is conjugate to (2.1):

$$-p_t - a_{01}p_x u - \frac{\partial}{\partial x}\int_x^\xi K(z-x,\alpha)\,p_z\,dz = \tag{3.4}$$
$$2[u(t,x;\alpha)-g(x)]\,\delta(t-\bar t),\quad p(\tau,x)=p(t,\xi)=0$$

Here $\delta(t-\bar t)$ is the delta-function, $p=p(t,x;\alpha)$, where $(t,x)\in D=(0,\tau)\times(0,\xi)$, $0<\bar t<\tau<\infty$.

We shall multiply both sides of Eq.(3.4) by $v_i(t,x;\alpha)$ and integrate them with respect to the domain D. Integrating by parts and using the boundary values of the functions v_i and p, we obtain

$$2\int_D [u(t,x;\alpha)-g(x)]\,v_i(t,x;\alpha)\,\delta(t-\bar t)\,dx\,dt = \frac{\partial J(\alpha)}{\partial\alpha_i} \tag{3.5}$$

$$-\int_D p_t v_i\,dx\,dt = \int_D (v_i)_t\,p\,dx\,dt \tag{3.6}$$

$$-\int_D p_x u v_i\,dx\,dt = \int_D (uv_i)_x\,p\,dx\,dt \tag{3.7}$$

$$-\int_0^\tau\int_0^\xi \frac{\partial}{\partial x}\int_x^\xi K(z-x,\alpha)\,p_z(t,z;\alpha)\,dz\,v_i(t,x;\alpha)\,dx\,dt = \tag{3.8}$$
$$-\int_0^\tau dt\int_0^\xi p(t,z;\alpha)\,\frac{\partial}{\partial z}\int_0^z K(z-x,\alpha)\,\frac{\partial v_i(t,x;\alpha)}{\partial x}\,dx\,dz$$

From relations (3.2), (3.5), (3.6), (3.7) and (3.8) it follows that

$$\frac{\partial J(\alpha)}{\partial\alpha_i} = \int_0^\tau\int_0^\xi \frac{\partial}{\partial x}\int_0^x \frac{\partial K(x-z,\alpha)}{\partial\alpha_i}\,u_z(t,z;\alpha)\,dz\,p(t,x;\alpha)\,dx\,dt \tag{3.9}$$

$i = 1,\ldots,n$

It is obvious from Eq.(3.9) that method (2.5) requires two integrodifferential equations at each step of the solution. Eqs.(2.1) and (3.4) are solved for $\alpha=\alpha^k$, then using Eq.(3.9) the gradient of the function $J(\alpha)$ is calculated and the new approximation α^{k+1} is found using Eq.(2.5), etc.

4. The parametric case when $K=K(x,\alpha)$ was considered above. The direction of the gradient in the space of the parameters $\alpha\in R_n$ was determined using Eq.(3.9), in which the solutions of Eqs.(2.1) and (3.4) occur. We could also directly use Eq.(3.3), which depends on the solutions of Eqs.(2.1) and (3.2). But we need to turn to an expression of the form (3.9) in the non-parametric case when the unknown function $K(x)$ is determined by observing $g(x)$.

For example, suppose $K(x)=K^0(x)-\alpha K^1(x)$, where $K^0(x)$ is a known function. If the correction $K^1(x)$ is considered to be known, the parametric case of the determination of the parameter α, described in the previous section, occurs. But it may turn out that precisely the form of the correction $K^1(x)$ is unknown and $K^1(x)$ must be determined using the measurement. Then the linear increase, relative to $K^1(x)$, of the functional $J(K)$ is obtained by transformations similar to those used when deriving Eq.(3.9).

Suppose L_2 is a class of quadratically integrable functions in the segement $[0,\xi]$. We put

$$(f_1, f_2) = \int_0^{\xi} f_1(x) f_2(x)\, dx$$

Suppose $u = u(t, x; K)$ is a solution of the equation

$$u_t + a_{01} u u_x - \frac{\partial}{\partial x} \int_0^x K(x - z) u_z\, dz = 0 \tag{4.1}$$

$$u(0, x) = u_0(x), \quad u(t, 0) = 0$$

We will assume that $K \in M \subset L_2$ where M is a convex compact set of smooth functions in L_2. We will not refine the properties of the function $K(x)$ and class M, and therefore the conclusions which follow below have a formal character. Suppose for each $K \in M$ the unique solution $u(t, x; K)$ of Eq.(4.1) exists, which has in the domain $D = (0, \tau) \times (0, \xi)$ the continuous derivatives u_t and u_x and which depends continuously on the function K together with the derivatives.

Suppose

$$J(K) = \int_0^{\xi} [u(\bar{t}, x; K) - g(x)]^2\, dx$$

It is required to determine $K^* \in M$, such that $J(K^*) = \min_{K \in M} J(K)$. Since M is a compact set, K^* exists.

We will assume that the gradient of the functional $J(K)$ exists. Then the method of projecting the gradient to calculate K^* has the form

$$K_{n+1} = P_M(K_n - t_n \operatorname{grad} J(K_n)), \quad n = 0, 1, \ldots \tag{4.2}$$

where $P_M(f)$ is the projection of the element f on the convex compact set M.

Suppose the function $p = p(t, x; K)$ satisfies Eq.(3.4), where the quantity $K(z - x, \alpha)$ is replaced by the function $K(z - x)$. We can then conclude (omitting the details) that

$$\left. \frac{dJ(K^0 + \alpha K^1)}{d\alpha} \right|_{\alpha=0} = (\operatorname{grad} J(K), K^1) = \int_0^{\tau} \int_0^{\xi} \frac{\partial}{\partial y} \int_y^{\xi} u_x(t, x - y; K)\, p(t, x; K)\, dx K^1(y)\, dy\, dt$$

Hence it is obvious that

$$\operatorname{grad} J(K) = \int_0^{\tau} \frac{\partial}{\partial y} \int_y^{\xi} u_x(t, x - y; K)\, p(t, x; K)\, dx\, dt$$

5. Suppose in Eq.(4.1) $a_{01} = 0$, i.e. we have the linear equation

$$u_t - \frac{\partial}{\partial x} \int_0^x K(x - z) u_z\, dz = 0; \quad u(0, x) = u_0(x), \quad u(t, 0) = 0 \tag{5.1}$$

where $u_0(0) = 0$.

Suppose $u_1(\bar{t}, x) = u(\bar{t}, x; K)$ is given, and it is required to determine the function $K(x)$.

We will apply Laplace's transformation using the argument x to Eq.(5.1).

Suppose

$$Lu = \bar{u}(t, s) = \int_0^{\infty} \exp(-sx)\, u(t, x)\, dx$$

$$Lu_0 = \bar{u}_0(s), \quad LK = \bar{K}(s), \quad Lu_1 = \bar{u}_1(s)$$

For the function $\bar{u}(t, s)$ which depends on parameter s we will obtain the equation

$$\bar{u}_t(t, s) - s^2 \bar{K}(s)\, \bar{u}(t, s) = 0, \quad \bar{u}(0, s) = \bar{u}_0(s)$$

whence $\bar{u}(t, s) = \bar{u}_0(s) \exp(s^2 \bar{K}(s) t)$. Since $\bar{u}(\bar{t}, s) = \bar{u}_1(s)$, then

$$\bar{K}(s) = \frac{1}{s^2 \bar{t}} \ln \frac{\bar{u}_1(s)}{\bar{u}_0(s)} \tag{5.2}$$

In view of the fact that $\bar{K}(s)$ does not depend on the $t = \bar{t}$, a function $A(s)$ must exist, such that the following condition holds:

$$\bar{u}(t, s) = \bar{u}_0(s) \exp(A(s) t) \tag{5.3}$$

Hence we obtain

$$\bar{K}(s) = A(s)/s^2 \tag{5.4}$$

Thus if condition (5.3) holds, where the function $A(s)$ is such that $A(s)/s^2$ is Laplace's transformation of the permissible functions $K \in M$, the function $u(\bar{t}, x) = g(x)$ uniquely defines the function $K(x)$ for any $\bar{t} > 0$.

It follows from Eq.(5.2) that a one-to-one and continuous mapping exists between the functions $K(x)$ and $u_1(\bar{t}, x) = u(\bar{t}, x; K)$. Since $K(x)$ belongs to the compact set M, the problem investigated is Tikhonov well-posed /4/.

It should be noted, however, that when $u_1(\bar{t}, x)$ is specified with an error, the problem of the membership of the calculated kernel K in the compact set M arises. It is quite clear that the error in $u_1(\bar{t}, x)$ can lead to a situation when the kernel $K(x)$ no longer belongs to the given compact set M. In that case we need to use Tikhonov's regularization method.

Eqs.(5.3) and (5.4) can serve as a solution of both Eq.(5.1) and the corresponding inverse problem.

Example. 1°. Suppose $\bar{u}_1(s) = \bar{u}_0(s) \exp \bar{t}$ (i.e. $u(t, x) = u_0(x) \exp t$). We will obtain from (5.3) that $A(s) \equiv 1$. Eq.(5.4) gives $\bar{K}(s) = 1/s^2$, whence $K(x) = x$. Eq.(5.1) is transformed to the form $u_t - u = 0$, whence, in fact, $u(t, x) = u_0(x) \exp t$.

Example. 2°. Suppose $K(x) = \alpha \exp(-\beta x)$ and the function $u_1(x) \approx u(\bar{t}, x; \alpha, \beta)$ is constructed using experimental data, on the basis of which it is required to determine the constants α and β. We will find $\bar{u}_1(s)$ and $\bar{K}(s) = \alpha/(s+\beta)$. From Eq.(5.4) we obtain that $A(s) = \alpha s^2/(s+\beta)$.

Then using (5.3)

$$\bar{u}_1(s) = \bar{u}_0(s) \exp(\alpha s^2 t/(s+\beta)) \tag{5.5}$$

We can use many methods to determine the approximate values of the parameters α and β From Eq.(5.5) (also bearing in mind the random errors when constructing the function $u_1(x)$, and thereby also $\bar{u}_1(s)$).

6. Suppose in Eq.(5.1) $K(x) = \alpha \exp(-\beta x)$. i.e.

$$u_t - \alpha \frac{\partial}{\partial x} \int_0^x \exp[-\beta(x-z)]\, u_z(t, z)\, dz = 0 \tag{6.1}$$

We will set

$$w(t, x) = \int_0^x \exp[-\beta(x-z)]\, u_z(t, z)\, dz$$

Since $w_x = u_x - \beta w$, we can replace Eq.(6.1) with the following equivalent system:

$$u_t - \alpha w_x = 0, \quad u_x - w_x - \beta w = 0$$
$$u(0, x) = u_0(x), \quad u(t, 0) = 0, \quad w(t, 0) = 0$$

Suppose $g(x) \approx u(\bar{t}, x; \alpha, \beta)$ is specified. It is required to determine α and β such that

$$J(\alpha, \beta) = \int_0^\xi [u(\bar{t}, x; \alpha, \beta) - g(x)]^2\, dx - \min \tag{6.2}$$

Consider the gradient method of determining the parameters α and β.
We will set $\partial u/\partial\alpha = u^1$, $\partial u/\partial\beta = u^2$, $\partial w/\partial\alpha = w^1$, $\partial w/\partial\beta = w^2$
We can show that

$$u_t^1 - \alpha w_x^1 = w_x, \quad u_x^1 - w_x^1 - \beta w^1 = 0$$
$$u^1(0, x) = w^1(t, 0) = u^1(t, 0) = 0$$

and

$$u_t^2 - \alpha w_x^2 = 0, \quad u_x^2 - w_x^2 - \beta w^2 = w$$
$$u^2(0, x) = w^2(t, 0) = u^2(t, 0) = 0$$

We will introduce the following conjugate systems:

$$-\frac{\partial p_1}{\partial t} + \frac{\partial q_1}{\partial x} = 2[u(t, x; \alpha, \beta) - g(x)]\,\delta(t - \bar{t})$$
$$-\frac{\partial q_1}{\partial x} + \alpha \frac{\partial p_1}{\partial x} + \beta q_1 = 0, \quad p_1(\tau, x) = p_1(t, \xi) = q_1(t, \xi) = 0$$
$$-\frac{\partial p_2}{\partial x} - \frac{\partial q_2}{\partial t} = 2[u(t, x; \alpha, \beta) - g(x)]\,\delta(t - \bar{t})$$
$$\alpha \frac{\partial q_2}{\partial x} + \frac{\partial p_2}{\partial x} - \beta p_2 = 0, \quad p_2(t, \xi) = q_2(\tau, x) = q_2(t, \xi) = 0$$

Using the arguments employed above, we can show that

$$\frac{\partial J(\alpha, \beta)}{\partial \alpha} = \int_0^\tau \int_0^\xi p_1(t, x; \alpha, \beta)\, w_x(t, x; \alpha, \beta)\, dx\, dt$$
$$\frac{\partial J(\alpha, \beta)}{\partial \beta} = \int_0^\tau \int_0^\xi p_2(t, x; \alpha, \beta)\, w(t, x; \alpha, \beta)\, dx\, dt$$

To determine the parameters α and β we have the following procedure:

$$\alpha_{n-1} = \alpha_n - s_n \frac{\partial J(\alpha_n, \beta_n)}{\partial \alpha}, \quad \beta_{n-1} = \beta_n - t_n \frac{\partial J(\alpha_n, \beta_n)}{\partial \beta}$$

where the quantitites s_n and t_n determine the length of the step of the gradient method.

7. The practical application of the methods discussed above encounter serious calculational difficulties. It is therefore best first of all to simplify the initial equation. We shall confine ourselves to one such simplification.

Suppose Eq.(6.1) is given. It is required to minimize the function (6.2).

We shall discretize Eq.(6.1) using Galerkin's method. Suppose $\{f_k(x)\}$ is a complete orthonormalized set of functions at the section $0 \leqslant x \leqslant \xi \leqslant \infty$. $f_k(0) = 0$. Suppose

$$u \approx u_N(t, x; \alpha, \beta) = \sum_{i=1}^{N} \varphi_i(t) f_i(x)$$

We shall substitute this expression into (6.1), multiply both sides of the equation by $f_k(x)$ and integrate it at the section $[0, \xi]$. Finally, we will obtain the system ($\varphi_k' = d\varphi_k/dt$) to determine the function $\varphi_k(t) = \varphi_k(t; \alpha, \beta)$

$$\varphi_k'(t) - \alpha \sum_{l=1}^{N} a_{kl}(\beta) \varphi_l(t) = 0, \quad \varphi_k(0) = c_k, \quad k = 1, \ldots, N \tag{7.1}$$

$$u_0(x) = \sum_{k=1}^{\infty} c_k f_k(x), \quad a_{kl}(\beta) = \int_0^{\xi} \frac{\partial}{\partial x} \int_0^{x} \exp(-\beta(x - z)) f_l'(z) (\partial z) f_k(x) \, dx$$

It is required to find α and β, such that they minimize the function

$$J_N(\alpha, \beta) = \int_0^{\xi} \Big[\sum_{k=1}^{N} \varphi_k(\bar{t}; \alpha, \beta) f_k(x) - g(x)\Big]^2 dx$$

The classical optimal control problem is obtained, to solve which there are well-developed and effective methods /7/. Note that, in principle, we can write the solution of system (7.1) analytically.

REFERENCES

1. NUNZIATO J.W., WALSH E.K., SCHULER K.W. and BARKER L.M., Wave propagation in non-linear viscoelastic solids. In: Handbuch der Physik. Ed. by C. Truesdell, B.: Springer, B. VIa/4, S., 1974.
2. ENGEL'BREKHT YU.K. and NIGUL U.K., Non-linear deformation waves. Moscow: Nauka, 1981.
3. ENGEL'BREKHT J., One-dimensional deformation waves in non-linear viscoelastic media. Wave Motion, 1, 1, 1979.
4. LAVRENT'EV M.M., ROMANOV V.G. and SHISHATSKII S.P., Ill-posed problems of mathematical physics and analysis. Moscow: Nauka, 1980.
5. WHITHAM G.B., Linear and non-linear waves. Moscow: Mir, 1977.
6. RUDENKO O.V. and SOLUYAN S.I., The theoretical principles of non-linear acoustics. Moscow: Nauka, 1975.
7. VASIL'EV F.P., Numerical methods of solving extremal problems. Moscow: Nauka, 1980.
8. LIONS J.L., Some aspects of modelling problems in distributed parameter systems. In: Lecture Notes in Control and Information Sciencies. B.: Springer. 1, 1978.

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